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Fuzzy Optimal Controller Design with Application to Fuzzy Modeled Structures

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Fuzzy analysis can be used to model the uncertainties present in structures. In particular, the fuzzy finite element method can be used to find the response of structures (isotropic or composite) involving uncertainties. For the integrated structural control of a system, the design of a fuzzy optimal controller is explored. In the past few years, some researchers have tried to design a fuzzy optimal controller based on the Takagi and Sugeno model. In this work, a different (new) approach is proposed for the design of a controller in the context of fuzzy optimal control theory. In most practical applications, the structural and material parameters vary considerably and are subject to uncertainties, mainly due to the uncontrollable aspects associated with the manufacturing and assembly of composite materials. These uncertainties, in many cases, cannot be modeled probabilistically because the probability distributions of the uncertain parameters are not usually known. Also, in some situations, the parameters are known only in linguistic form. A fuzzy finite element approach has been developed for the analysis of structures with fuzzy parameters. Using the deterministic optimal control theory as a basis, a fuzzy optimal control theory is developed so that the controller can be used to act as either a regulator or a tracking system. The control of a planar truss is presented to demonstrate the feasibility and applicability of the methodology presented.

I. Introduction

FUZZY logic control has been used in commercial electronics for the last several years. In these applications, if–then rules are often used to formulate the conditional statements. A fuzzy if–then rule is of the form "if x is A then y is B," where A and B may be fuzzy variables defined by fuzzy sets on the ranges of X and Y, respectively. For a simple fuzzy control, "if x is A" can be treated as some condition on the state variables of the system or other conditions on the system, whereas "if y is B" can be interpreted as some actuator force on the control system. Because crisp values are available from sensors and crisp values are required for actuation, the so-called fuzzification and defuzzification algorithms need to be applied to assign fuzzy set memberships to signal values and vice versa. In fact, if–then fuzzy rules can only be used for simple control systems such as water level control in a tank, temperature control in a room, etc.

When complicated dynamic linear/nonlinear systems are considered, the fuzzy variables x and y lack the ability to model the complex system. Hence, different fuzzy approaches and models can be used in fuzzy control theory. Takagi and Sugeno [1] presented a mathematical tool to build a fuzzy model of a system, which is called the Takagi–Sugeno fuzzy system model. It is based on a fuzzy partition of the input space. In each fuzzy subspace, a linear input–output relation is formed. The output of fuzzy reasoning is given by the aggregation of the values inferred by some implications that were applied to an input. Based on this model, many researchers presented their own fuzzy control theories. Because stability and optimality are the most important requirements for any control system, a number of researchers conducted research on optimal fuzzy control recently, although the area still remains quite unexplored. Wang [2] presented a number of stable and optimal fuzzy controllers applied to a linear ball-and-beam system. Zak [3] used a Lyapunov-based approach to derive a sufficient condition for stabilizing a class of Takagi–Sugeno fuzzy system models using linear controllers, and a system consisting of an inverted pendulum mounted on a cart was used to verify the method. Ma et al. [4] considered the analysis and design of a fuzzy controller and fuzzy observers on the basis of the Takagi–Sugeno fuzzy system model and the separation property. The numerical simulation and experiment on an inverted pendulum system were given to illustrate the performance of fuzzy controllers and fuzzy observers. Tanaka et al. [5,6] presented the stability analysis of a class of uncertain nonlinear systems and a method of designing robust fuzzy controllers to stabilize the uncertain nonlinear systems. Wu and Lin [7,8] presented two approaches for the optimal fuzzy controller design; one was a local concept approach, and the other was a global concept approach for continuous fuzzy systems. The local one presented a globally optimal and stable fuzzy controller design method for both continuous- and discrete-time fuzzy systems under both a finite and infinite horizon (time). The global concept approach proposed a systematic and theoretically sound way to design a globally optimal fuzzy controller to control and stabilize a continuous fuzzy system with a free- or fixed-end point under a finite or infinite horizon (time). The optimal fuzzy control can be very useful in linear/nonlinear fuzzy system control.

When a complex structure involving fuzzy uncertainties is considered, the fuzzy finite element analysis is often used to model the structure (not the Takagi–Sugeno model, but rather a new model). To control such a fuzzy modeled structure and to let the structure operate in a dynamic environment, a suitable method needs to be developed to control the dynamic response of the structure. In this paper, a new fuzzy optimal control method, based on fuzzy sets, fuzzy structural models, and the linear quadratic regulator (LQR) optimal control theory [9], is presented for controlling structures (both isotropic and composite ones) involving fuzzy parameters. The stability of the fuzzy optimal control method is also investigated. The goal of designing a control system is to achieve an optimal and stable system. Further, with the proposed fuzzy optimal control, possible structural health monitoring can be achieved.

II. Fuzzy Sets

A fuzzy set can be represented as [10]

$$F = \{(x, \mu_F(x)) | x \in U\}, \quad \text{where } \alpha = \mu_F\colon U \to [0, 1] \tag{1}$$

A fuzzy number is defined as a normal and convex fuzzy set, whereas a fuzzy set F is considered convex if and only if $\forall\, x^{(1)}, x^{(2)} \in R$, $\forall\, \lambda \in [0, 1]$,

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$$\mu_F[\lambda x^{(1)} + (1-\lambda)x^{(2)}] \geq \mu_F(x^{(1)}) \wedge \mu_F(x^{(2)}) \tag{2}$$

where $X \wedge Y$ means the minimum of X and Y; a fuzzy set is defined to be normal if and only if

$$\forall\ x \in R\colon\ \vee_n\ \mu_F(x) = 1 \tag{3}$$

where $X \vee Y$ means the maximum of X and Y, and n is the total number of membership functions of F. This implies that the highest value of $\mu_F(x)$ is equal to 1. This maximum may or may not be unique. In this work, the symbols $(+)$, $(-)$, $(.)$, and $(/)$ are used to denote fuzzy number addition, subtraction, multiplication, and division by max–min convolution.

III. Fuzzy Structural Model

The fuzzy finite element model of a structure leads to the global stiffness and mass matrices of the system, $[K]$ and $[M]$, where the elements of the $n \cdot n$ matrices are fuzzy numbers that are described by the α-cut approach with known membership functions. These two matrices can be used to describe the equations of motion of the structure. For the forced vibration of a structure with active controls, the coupled differentiation equations corresponding to the fuzzy finite element model can be expressed as

$$[M(d)](\cdot)\ddot{r}(t)(+)[C(d)](\cdot)\dot{r}(t)(+)[K(d)](\cdot)r(t) = F_1(u,t)(+)F_2(t) \tag{4}$$

where n denotes the number of degrees of freedom of the system; the $n \cdot 1$ vector $r(t)$ represents the system response in the configuration space; and the $n \cdot n$ mass, damping, and stiffness matrices, $[M]$, $[C]$, and $[K]$, are functions of the structural variables d. The number of structural variables can be as few as one or as many as an integral number of finite elements in the structure. Although the mass matrix generally consists of both structural and nonstructural components, only the structural mass will be influenced by the structural variables. The crisp stiffness and mass matrices, $[K]$ and $[M]$, are symmetric and positive definite, or at least semidefinite. If there is no damping, $C = [(0)]$; otherwise, the damping matrix is assumed to be proportional to the stiffness and mass matrices as

$$[C(d)] = a_1(\cdot)[K(d)](+)a_2(\cdot)[M(d)] \tag{5}$$

where a_1 and a_2 are constants. In Eq. (4), $F_1(u,t)$ represents the control input and $F_2(t)$ denotes the external disturbances. If only free vibration is considered, $F_2(t) = (0)$ and Eq. (4) reduces to

$$[M(d)](\cdot)\ddot{r}(t)(+)[C(d)](\cdot)\dot{r}(t)(+)[K(d)](\cdot)r(t) = b(\cdot)u(t) \tag{6}$$

It is assumed that the control system consists of a set of m discrete actuators. The $m \cdot 1$ vector $u(t)$ represents the input of the actuators, whereas b is an $n \cdot m$ matrix that identifies the position and the relationship between the controllers and the actuators. Because Eq. (6) represents a fuzzy model, the corresponding optimal control rule to control the structure should be fuzzy and stable. A new fuzzy optimal control strategy is developed in this work. A typical membership function of the elements of the matrices $[K]$, $[M]$, and $[C]$ is assumed to be of the form shown in Fig. 1. In numerical computations, several discrete α cuts are used in the range of $\alpha_0 = 0.0$ to $\alpha_5 = 1.0$ in increments of 0.2. Thus, x_j denotes the crisp value of the matrix element that corresponds to α_j, with x_j^- ($j = 0, 1, 2, 3, 4, 5$) indicating the smaller value and x_j^+ representing the larger value of the matrix element corresponding to α_j. The fuzzy element can be stated in discrete form as $x = \{x_i/\mu(x_i)\}$, where $i = 1, 2, \ldots, 11$, $\quad x_i = \{x_0^-, x_1^-, x_2^-, x_3^-, x_4^-, x_5, x_4^+, x_3^+, x_2^+, x_1^+, x_0^+\}$, and $\mu(x_i) = \mu_i = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0\}$. Although the number of discrete values is chosen as 11 (the corresponding number of α cuts is 6) in this work, it can be taken as equal to any other integer N.

For each μ_i, there corresponds a deterministic value x_i and, hence, for each i, Eq. (6) can be written in the deterministic form

$$[M_i(d)]\ddot{r}_i(t) + [C_i(d)]\dot{r}_i(t) + [K_i(d)]r_i(t) = b_i u_i(t) \tag{7}$$

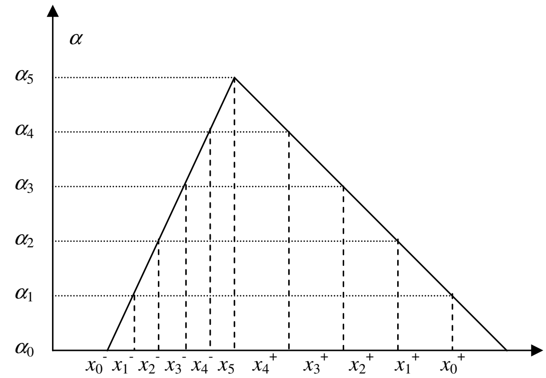

Fig. 1 Membership function of a typical element of a fuzzy matrix.

where $[M_i]$, $[C_i]$, and $[K_i]$ are deterministic matrices with each element corresponding to μ_i; and r_i, b_i, and $u_i(t)$ are deterministic vectors with each element corresponding to μ_i; $i = 1, 2, \ldots, N$. For illustration, we consider a system with just 2 degrees of freedom ($n = 2$) with the damping matrix $[C] = [(0)]$. Let $\alpha = \{0.0^-, 1.0, 0.0^+\}$, so that $\alpha_{i=1} = 0.0^-$, $\alpha_{i=2} = 1.0$, and $\alpha_{i=3} = 0.0^+$. Then the matrices and vectors of Eq. (7) can be expressed as

$$M = \begin{bmatrix} \begin{Bmatrix} M_{11,\alpha=0.0^-} \\ M_{11,\alpha=1.0} \\ M_{11,\alpha=0.0^+} \end{Bmatrix} & \begin{Bmatrix} M_{12,\alpha=0.0^-} \\ M_{12,\alpha=1.0} \\ M_{12,\alpha=0.0^+} \end{Bmatrix} \\ \begin{Bmatrix} M_{21,\alpha=0.0^-} \\ M_{21,\alpha=1.0} \\ M_{21,\alpha=0.0^+} \end{Bmatrix} & \begin{Bmatrix} M_{22,\alpha=0.0^-} \\ M_{22,\alpha=1.0} \\ M_{22,\alpha=0.0^+} \end{Bmatrix} \end{bmatrix}_{2\cdot 2},$$

$$K = \begin{bmatrix} \begin{Bmatrix} K_{11,\alpha=0.0^-} \\ K_{11,\alpha=1.0} \\ K_{11,\alpha=0.0^+} \end{Bmatrix} & \begin{Bmatrix} K_{12,\alpha=0.0^-} \\ K_{12,\alpha=1.0} \\ K_{12,\alpha=0.0^+} \end{Bmatrix} \\ \begin{Bmatrix} K_{21,\alpha=0.0^-} \\ K_{21,\alpha=1.0} \\ K_{21,\alpha=0.0^+} \end{Bmatrix} & \begin{Bmatrix} K_{22,\alpha=0.0^-} \\ K_{22,\alpha=1.0} \\ K_{22,\alpha=0.0^+} \end{Bmatrix} \end{bmatrix}_{2\cdot 2}$$

$$r(t) = \begin{bmatrix} \begin{Bmatrix} r_{1,\alpha=0.0^-} \\ r_{1,\alpha=1.0} \\ r_{1,\alpha=0.0^+} \end{Bmatrix} \\ \begin{Bmatrix} r_{2,\alpha=0.0^-} \\ r_{2,\alpha=1.0} \\ r_{2,\alpha=0.0^+} \end{Bmatrix} \end{bmatrix}_{2\cdot 1}, \tag{8}$$

$$b = \begin{bmatrix} \begin{Bmatrix} b_{11,\alpha=0.0^-} \\ b_{11,\alpha=1.0} \\ b_{11,\alpha=0.0^+} \end{Bmatrix} & \begin{Bmatrix} b_{12,\alpha=0.0^-} \\ b_{12,\alpha=1.0} \\ b_{12,\alpha=0.0^+} \end{Bmatrix} \\ \begin{Bmatrix} b_{21,\alpha=0.0^-} \\ b_{21,\alpha=1.0} \\ b_{21,\alpha=0.0^+} \end{Bmatrix} & \begin{Bmatrix} b_{22,\alpha=0.0^-} \\ b_{22,\alpha=1.0} \\ b_{22,\alpha=0.0^+} \end{Bmatrix} \end{bmatrix}_{2\cdot 2},$$

$$u(t) = \begin{bmatrix} \begin{Bmatrix} u_{1,\alpha=0.0^-} \\ u_{1,\alpha=1.0} \\ u_{1,\alpha=0.0^+} \end{Bmatrix} \\ \begin{Bmatrix} u_{2,\alpha=0.0^-} \\ u_{2,\alpha=1.0} \\ u_{2,\alpha=0.0^+} \end{Bmatrix} \end{bmatrix}_{2\cdot 1}$$

For each μ_i, $i = 1, 2, 3$, Eq. (7) can be written in a deterministic form as

$$\begin{bmatrix} M_{11,\alpha_i} & M_{12,\alpha_i} \\ M_{21,\alpha_i} & M_{22,\alpha_i} \end{bmatrix} \begin{bmatrix} \ddot{r}_{1,\alpha_i} \\ \ddot{r}_{2,\alpha_i} \end{bmatrix} + \begin{bmatrix} K_{11,\alpha_i} & K_{12,\alpha_i} \\ K_{21,\alpha_i} & K_{22,\alpha_i} \end{bmatrix} \begin{bmatrix} r_{1,\alpha_i} \\ r_{2,\alpha_i} \end{bmatrix} = \begin{bmatrix} b_{11,\alpha_i} & b_{12,\alpha_i} \\ b_{21,\alpha_i} & b_{22,\alpha_i} \end{bmatrix} \begin{bmatrix} u_{1,\alpha_i} \\ u_{2,\alpha_i} \end{bmatrix} \tag{9}$$

For each μ_i, the deterministic open-loop state-space equations corresponding to Eq. (7) can be written as

$$\dot{X}_i = A_i X_i + B_i u_i \tag{10}$$

$$\dot{X}_i = \begin{bmatrix} \ddot{r}_i \\ \dot{r}_i \end{bmatrix}_{2n\cdot 1} \qquad X_i = \begin{bmatrix} \dot{r}_i \\ r_i \end{bmatrix}_{2n\cdot 1} \tag{11}$$

where the open-loop plant matrix A_i and the control matrix B_i are given by

$$A_i = \begin{bmatrix} -M_i^{-1}C_i & -M_i^{-1}K_i \\ [I] & [0] \end{bmatrix}_{2n\cdot 2n} \qquad B_i = \begin{bmatrix} M_i^{-1}b_i \\ 0 \end{bmatrix}_{2n\cdot m} \tag{12}$$

with $[I]$ representing an $n \cdot n$ unity matrix and $[0]$ denoting an $n \cdot n$ zero matrix. Equation (10) constitutes the ith subsystem and, in view of the matrices defined in Eq. (12), denotes the full system formulation without any modal reduction. If modal reduction is used with

$$r_i(t) = [\phi_i]\eta_i(t) \tag{13}$$

where the $p \cdot 1$ vector $\eta_i(t)$ represents the normal coordinates and the $n \cdot p$ matrix $[\phi_i]$ denotes the modal matrix obtained from the solution of the eigenvalue problem,

$$\omega^2[M_i]r_i = [K_i]r_i \tag{14}$$

where ω^2 represents the eigenvalue of the system. The number of modes p to be retained ($p \leq n$) to represent the dynamic response depends on the type of disturbance and the number and locations of the actuators and sensors. When modal reduction is used, the state vectors in Eq. (11) are given by

$$\dot{X}_i = \begin{bmatrix} \ddot{\eta}_i \\ \dot{\eta}_i \end{bmatrix}_{2p\cdot 1} \qquad X_i = \begin{bmatrix} \dot{\eta}_i \\ \eta_i \end{bmatrix}_{2p\cdot 1} \tag{15}$$

and the plant and control matrices of the open-loop system are given by

$$A_i = \begin{bmatrix} [-2\varsigma_j\omega_j] & [-\omega_j^2] \\ [I] & [0] \end{bmatrix}_{2p\cdot 2p}, \qquad B_i = \begin{bmatrix} \phi_i^T b_i \\ 0 \end{bmatrix}_{2p\cdot m} \tag{16}$$

where $[I]$ represents a $p \cdot p$ unity matrix and $[0]$ denotes a $p \cdot p$ zero matrix. The submatrices in A_i are all diagonal matrices. The modal damping ratio ς_j is given by

$$\varsigma_j = (a_1/(2/\omega_j)) + (a_2/2\omega_j) \tag{17}$$

where a_1 and a_2 are constants. The underdamped case, with the damping ratio $\varsigma_j < 1$, is of interest in the present work. The output of the system can be represented by

$$[y_i]_{s\cdot 1} = [C_y]_{s\cdot 2n}[X_i]_{2n\cdot 1} \tag{18}$$

which implies the inclusion of both velocity and displacement sensors, and the case of $s = 2n$ represents the situation in which there are enough sensors to measure all of the states of the system. In most practical cases, however, $s < 2n$, and the matrix C_y, in such a case, contains only zeros and ones with the ones denoting the state variables along which the sensors are located. From the analysis presented, a new fuzzy model can be represented as follows.

When all of the elements of the fuzzy matrices and vectors in the governing differential equations of the system correspond to the same α cut, the values of the fuzzy numbers correspond to the same μ_i, $i = 1, 2, \ldots, N$. Then the fuzzy system model is defined by rule i: If the α cut of the matrices and the vectors is equal to μ_i, then

$$\dot{X}_i = A_i X_i + B_i u_i, \qquad i = 1, 2, \ldots, N \qquad y_i = C_y X_i \tag{19}$$

where the state vector $X_i = [x_{i1}, x_{i2}, \ldots, x_{i,2n}]^T \in R^{2n}$, the system output $y_i \in R^s$, the system input (control output) $u_i \in R^m$. A_i is a $2n \cdot 2n$ matrix, B_i is a $2n \cdot m$ matrix, and C_y is a $s \cdot 2n$ matrix. The elements of all of these matrices and vectors correspond to μ_i. In this work, this fuzzy model is used to construct the real structural model, and the fuzzy optimal control theory is developed based on this model.

IV. Fuzzy Optimal Control Theory

A. Fuzzy Optimal Control Rule

To explore the fuzzy optimal control theory of the fuzzy model indicated in Eq. (19), the following proposition is made: the global optimal effect can be achieved by the fuzzily combined local optimal controllers and, based on this, a fuzzy optimal control theory can be derived by applying the traditional linear optimal control theory. The control variables are given by the $m \cdot 1$ vector $u(t)$, which represents the inputs from the m actuators. According to modern control theory, a quadratic performance index J, which is a function of the state and the control vectors $X(t)$ and $u(t)$, will be minimized:

$$\begin{aligned} J &= \sum_{i=1}^{N} h_i[X^T(t_f)(\cdot)H_i(\cdot)X(t_f)] \\ &\quad + \sum_{i=1}^{N} h_i\left\{\int_0^{t_f}[X^T(\cdot)Q_i(\cdot)X(+)u^T(\cdot)R_i(\cdot)u]\,\mathrm{d}t\right\} \\ h_i(V) &= \frac{\mu(V_i)V_i}{\sum_{j=1}^{N}\mu(V_j)}, \\ h_i[V(\Box)W] &= h_i(V)\Box h_i(W), (\Box) = (+), (-), (\cdot) \end{aligned} \tag{20}$$

where H_i, Q_i, and R_i are deterministic weighting matrices (R_i must be positive definite, Q_i must at least be positive semidefinite), X and u are fuzzy vectors, V and W are fuzzy numbers with V_i denoting the value of the fuzzy number V corresponding to $\mu(V_i)$, N is the number of α-cut values of the element of any fuzzy vector (N will be same for different fuzzy numbers), and $h_i(V)$ can be called a fuzzy operator of a fuzzy number V. From Eq. (20), it can be shown that J will be evaluated from time 0 to t_f and will be a minimum when $u = u^*$, which is the fuzzy optimal control force. The control rule of the fuzzy model can be stated as follows.

Rule i: If the α cut of the matrices and the vectors is equal to μ_i, then

$$u_i^*(t) = -G_i X_i^*(t) = -R_i^{-1}B_i^T P_i(t) X_i^*(t), \qquad i = 1, 2, \ldots, N \tag{21}$$

It is obvious that the ith subsystem and the ith fuzzy control rule have a one-to-one mapping. Therefore, the global control rule can be derived by fuzzily combining each subsystem control rule u_i^* corresponding to different μ_i. The matrices Q_i or R_i may be chosen to be the same for different i, or different Q_i and R_i may be selected for different i. u_i^* denotes the optimal control force and X_i^* indicates the optimal state vector of the ith subsystem. G_i is called the control gain matrix of the ith subsystem and is given by $G_i = R_i^{-1}B_i^T P_i$, and $P_i(t)$ is the solution of the finite time Riccati equation:

$$\begin{aligned} \dot{P}_i(t) &= -A_i^T P_i(t) + P_i(t)B_i R_i^{-1}B_i^T P_i(t) - P_i(t)A_i - Q_i, \\ P_i(t_f) &= H \end{aligned} \tag{22}$$

Alternatively, P_i can be obtained as a symmetric positive definite matrix from the solution of the infinite time ($t_f = \infty$ and $X_i(t_f) = [0]$) algebraic Riccati equation in a time invariant system as

$$0 = A_i^T P_i - P_i B_i R_i^{-1} B_i^T P_i + P_i A_i + Q_i \tag{23}$$

The fuzzy system actuator force u^* will be obtained by combining each subsystem control force u_i^* as

$$u^* = \{u_i^*/\mu_i\}, \qquad i = 1, 2, \ldots, N \tag{24}$$

The optimal ith closed-loop subsystem will be given by

$$\dot{X}_i^*(t) = [A_i - R_i^{-1}B_i^T P_i]X_i^*(t) \tag{25}$$

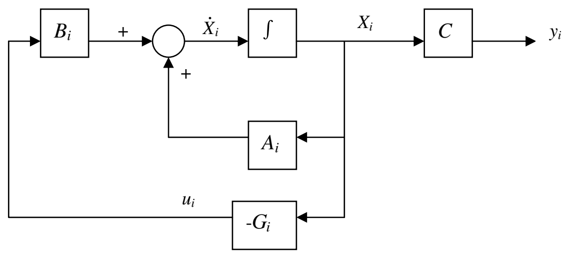

Fig. 2 Optimal control subsystem.

From this analysis, it can be noted that, for each μ_i, the deterministic linear optimal control theory is used to obtain the optimal control force u_i^*, which is then used to formulate the final fuzzy optimal control force by combining the individual local linear optimal control forces for the fuzzy model represented by Eq. (19). The control subsystem is shown in Fig. 2.

This procedure can be generalized in the form of the following theorem.

Theorem 1: Solution of the Fuzzy Optimal Control Problem: For a fuzzy system consisting of a series of subsystems governed by Eq. (19) and a fuzzy controller given by Eq. (24), let $A_i(t)$ and $B_i(t)$ be the matrices corresponding to μ_i, and Q_i and R_i be at least positive semidefinite symmetric matrices. If there exists on [0, t_f] a symmetric positive semidefinite solution $P_i(t)$ to the matrix Riccati differential equation shown in Eq. (22) with $t \in [0, t_f]$, then there exists a local optimal control law $u_i^*(t) = -G_i X_i^*(t)$ with $G_i = R_i^{-1} B_i^T P_i$, where $X_i^*(t)$ is the corresponding optimal state trajectory. Then $u^* = \{u_i^*/\mu_i\}$, $i = 1, 2, \ldots, N$, minimizes $J(u(\cdot))$ given in Eq. (20). The resulting optimal closed-loop system dynamics can be represented as

$$\dot{X}^* = \sum_{i=1}^{N} h_i(A_i) X^*(t) - \sum_{i=1}^{N} h_i[R_i^{-1} B_i^T P_i] X^*(t), \qquad t \in [0, t_f],$$
$$X(0) = X_0 \qquad h_i(U) = [\,h_i(U_1) \quad h_i(U_2) \quad \cdots \quad h_i(U_n)\,]^T,$$
$$h_i(D) = [h_i(D_{jk})] \tag{26}$$

where U denotes a fuzzy vector with U_i indicating the ith element of U, and D denotes a fuzzy matrix with D_{jk} representing the fuzzy element lying on the jth row and kth column.

Proof: The cost function shown in Eq. (20) is

$$J = \sum_{i=1}^{N} h_i[X^T(t_f)(\cdot) H_i(\cdot) X(t_f)]$$
$$+ \sum_{i=1}^{N} h_i \left\{ \int_0^{t_f} [X^T(\cdot) Q_i(\cdot) X(+) u^T(\cdot) R_i(\cdot) u] \, \mathrm{d}t \right\}$$
$$h_i(V) = \frac{\mu(V_i) V_i}{\sum_{j=1}^{N} \mu(V_j)}, \qquad h_i[V(\Box)W] = h_i(V) \Box h_i(W),$$
$$(\Box) = (+), (-), (\cdot) \tag{27}$$

For the ith subsystem, the LQR theory ensures that u_i^* minimizes the following cost function:$J_i = \int_0^{t_f} [X_i^T Q_i X_i + u_i^T R_i u_i] \, \mathrm{d}t$ and $J_i^* = \int_0^{t_f} [X_i^{*T} Q_i X_i^* + u_i^{*T} R_i u_i^*] \, \mathrm{d}t$. Because H_i, Q_i, and R_i are positive matrices, and $\mu_i \geq 0$, $i = 1, 2, \ldots, N$, we can obtain

$$\sum_{i=1}^{N} h_i[u^T(\cdot)u] = \sum_{i=1}^{N} h_i \left([\, u_1 \quad \cdots \quad u_m \,] \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \right)$$
$$= \sum_{i=1}^{N} h_i \{ [u_1(\cdot)u_1](+) \cdots (+)[u_m(\cdot)u_m] \}$$
$$= \frac{1}{(\sum_{j=1}^{N} \mu_j)^2} \left\{ \sum_{i=1}^{N} \left[\mu_i^2 \sum_{k=1}^{m} (u_k)_i^2 \right] \right\} \tag{28}$$

When $u = u^* = \{u_i^*/\mu_i\}$, J attains its minimum. Thus, the theorem holds true.

When time varies from 0 to ∞, for the deterministic LQR problem, when the system is time invariant and completely controllable and observable, there exists a solution. Suppose the fuzzy system under consideration is also time invariant and completely controllable and observable; then, the following theorem holds true.

Theorem 2: For a fuzzy system consisting of a series of subsystems governed by Eq. (19) and a fuzzy controller given by Eq. (24), let A_i and B_i be the matrices corresponding to μ_i, Q_i and R_i be at least positive semidefinite symmetric matrices, and $Q_i = C^T C$. If (A_i, B_i) is completely controllable and (A_i, C) is completely observable, then there exists a symmetric positive semidefinite solution P_i to the matrix algebraic Riccati equation shown in Eq. (23) and a local optimal control law $u_i^*(t) = -G_i X_i^*(t)$ with $G_i = R_i^{-1} B_i^T P_i$, where $X_i^*(t)$ is the corresponding optimal state trajectory and $u^* = \{u_i^*/\mu_i\}$, $i = 1, 2, \ldots, N$, which minimizes $J(u(\cdot))$ given by

$$J = \sum_{i=1}^{N} \left[h_i \left\{ \int_0^{\infty} X^T(\cdot) Q_i(\cdot) X(+) u^T(\cdot) R_i(\cdot) u \right] \mathrm{d}t \right\} \tag{29}$$

The resulting optimal closed-loop system dynamics can be described by

$$\dot{X}^* = \sum_{i=1}^{N} h_i(A_i) X^*(t) - \sum_{i=1}^{N} h_i \left[R_i^{-1} B_i^T P_i \right] X^*(t), \qquad X(0) = X_0$$
$$h_i(U) = [\,h_i(U_1) \quad h_i(U_2) \quad \cdots \quad h_i(U_n)\,]^T, \qquad h_i(D) = [h_i(D_{jk})] \tag{30}$$

For each subsystem, $X_i^*(t)$ is asymptotically and exponentially stable, which means that $X_i^*(t)$ tends to 0 as $t \to \infty$.

Proof: This theorem obviously holds based on the deterministic LQR theory, and a proof similar to that of Theorem 1 can be given.

B. Tracking System Design

In structural health monitoring, when the structure operates normally with no damage, the system displacements and velocities are assumed to be equal to some ideal (reference) values. These ideal values may be recorded as the expected and desirable values of the control system. If there is any damage to the structure, the displacements and velocities read from the sensors will be different from the previously recorded values. Under these critical (damaged) conditions, a new optimal controller should be designed to maintain the values of the displacements and velocities equal to the expected values (or at least equal to values within a reasonable range of the expected values) within a specified finite time period. During the limited time period, the damaged structure can give a signal or alarm, and the operator can take measures to maintain the safety and health of the structure. This process denotes the scheme of maintaining the health of the structure. The system can now be considered a tracking system that tracks the expected values of the displacements and velocities of the structure using fuzzy optimal control. Because the fuzzy optimal control is achieved by fuzzily combining a series of linear LQR controllers, a series of tracking subsystems based on linear LQR can be designed [9].

In the tracking system design, integral control is used to zero out the steady-state errors when tracking constant signals (expected values). Integral control can be generated in an LQR setting by appending an integrator to the plant before computing the feedback gain G_i. The integral of the error between the reference (expected) input rr and the output y_i is generated by the following differential equation:

$$\dot{e}_{Ii}(t) = rr - y_i(t) = -C_y X_i(t) + rr \tag{31}$$

The error in Eq. (31) needs to be integrated for each of the reference inputs that is being tracked (to zero out the errors), which makes the number of integrators equal to the number of reference inputs. The augmented state model is a combination of the state equation of the plant, shown in Eq. (19), and the state equation of the integral of the error, shown in Eq. (31). Thus, the augmented state equation can be expressed as

$$[\dot{e}_i(t)] = \begin{bmatrix} \dot{X}_i(t) \\ \dot{e}_{Ii}(t) \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ -C_y & 0 \end{bmatrix} \begin{bmatrix} X_i(t) \\ e_{Ii}(t) \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} rr \tag{32}$$

The LQR theory can be used to generate a state feedback for the augmented plant. This controller is designed ignoring the constant reference input and using a cost function that penalizes the integral of the error as

$$\begin{aligned} J_{Ii} &= \int_0^\infty \left[e_i^T(t) e_i(t) + u_i^T(t) R_i u_i(t) \right] \mathrm{d}t \\ &= \int_0^\infty \left\{ \begin{bmatrix} X_i^T(t) & e_{Ii}^T(t) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} X_i(t) \\ e_{Ii}(t) \end{bmatrix} + u_i^T(t) R_i u_i(t) \right\} \mathrm{d}t \end{aligned} \tag{33}$$

When LQR theory is used to obtain the gain matrix G_i, the optimal control is given by

$$\begin{aligned} u_i(t) &= -G_i(t) \begin{bmatrix} X_i(t) \\ e_{Ii}(t) \end{bmatrix} = -[\, G_{ix}(t) \quad G_{iI}(t)\,] \begin{bmatrix} X_i(t) \\ e_{Ii}(t) \end{bmatrix} \\ &= -G_{ix}(t) X_i(t) - G_{iI}(t) \int_0^\infty [rr - y_i(t)]\, \mathrm{d}t \end{aligned} \tag{34}$$

which shows that the control includes integral feedback. The fuzzy optimal control law of the fuzzy tracking system will be the same as the one given in Eq. (24): $u^* = \{u_i^*/\mu_i\}$, $i = 1, 2, \ldots, N$. The integral control tracking subsystem is shown in Fig. 3.

C. System Simulation Using Defuzzification

Before applying the actual (fuzzy) force to the real structure, the fuzzy force is to be defuzzified to obtain an equivalent crisp value as

$$\bar{u}^*[j] = \frac{\sum_{i=1}^N \mu_i u_i^*[j]}{\sum_{i=1}^N \mu_i}, \qquad j = 1, 2, \ldots, m \tag{35}$$

Because each $u_i^*[j]$ is treated as a value of the fuzzy number $u^*[j]$, a fuzzy number truncation process [6] may need to be applied here. The elements of the fuzzy matrices A and B are also to be defuzzified as

$$\bar{A}_{ij} = \frac{\sum_{k=1}^N \mu_k (A_{ij})_k}{\sum_{i=k}^N \mu_k}, \qquad \bar{B}_{ij} = \frac{\sum_{k=1}^N \mu_k (B_{ij})_k}{\sum_{i=k}^N \mu_k}, \qquad i, j = 1, 2, \ldots, 2n \tag{36}$$

where $\bar{A}_{ij}$ and $\bar{B}_{ij}$ are the deterministic elements lying in the ith row and jth column of the matrices $\bar{A}$ and $\bar{B}$, respectively, and A_{ij} and B_{ij} are the fuzzy elements lying in the ith row and jth column of the matrices A and B, respectively. In the system configuration shown in Figs. 2 and 3, A_i, B_i, and u_i are replaced by $\bar{A}$, $\bar{B}$, and $\bar{u}^*$, respectively, to simulate the system to find the effects of control.

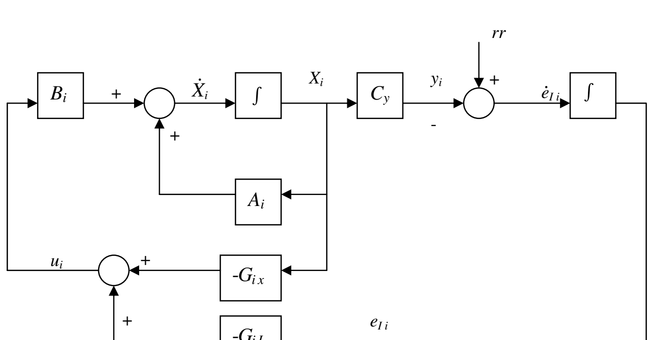

Fig. 3 Tracking subsystem with optimal control.

D. Stability of the System

Time Invariant System

We will only consider the stability of the time invariant system here. The fuzzy optimal control rule can be expressed as

$$u^* = -\sum_{i=1}^N h_i(G_i) X^*(t), \qquad G_i = R_i^{-1} B_i^T P_i \tag{37}$$

For the closed-loop fuzzy optimal control system, the following equation can be given:

$$\begin{aligned} \dot{X}^* &= \sum_{i=1}^N h_i(A_i) X^*(t) + \sum_{i=1}^N h_i(B_i) u^*(t) = \sum_{i=1}^N h_i(A_i) X^*(t) \\ &\quad - \sum_{i=1}^N h_i(B_i) \sum_{j=1}^N h_j(G_j) X^*(t) \end{aligned} \tag{38}$$

Because

$$\sum_{i=1}^N h_i[(1)] = \frac{\sum_{i=1}^N \mu_i(1)_i}{\sum_{i=1}^N \mu_i} = 1$$

Eq. (38) can be rewritten as

$$\begin{aligned} \dot{X}^* &= \sum_{i=1}^N h_i(A_i) X^*(t) + \sum_{i=1}^N h_i(B_i) u^*(t) \\ &= \sum_{i=1}^N h_i(A_i) \sum_{j=1}^N h_j[(1)] X^*(t) - \sum_{i=1}^N h_i(B_i) \sum_{j=1}^N h_j(G_j) X^*(t) \\ &= \sum_{i=1}^N \sum_{j=1}^N h_i h_j [A_i - B_i G_j] X^*(t) \end{aligned} \tag{39}$$

From Lyapunov's direct method [11], for a system $\dot{X}(t) = F(X(t))$, the satisfaction of the sufficient condition guarantees the stability of the fuzzy system, that is, $F(X_o) = 0$, where X_o denotes the equilibrium state. In the present case, Eq. (39) is asymptotically stable in the large if there exists a common positive definite matrix P_s such that

$$(A_i - B_i G_i)^T P_s + P_s (A_i - B_i G_i) < 0 \tag{40}$$

for $i = 1, 2, \ldots, N$, and

$$\left(\frac{A_i - B_i G_j + A_j - B_j G_i}{2} \right)^T P_s + P_s \left(\frac{A_i - B_i G_j + A_j - B_j G_i}{2} \right)^T < 0 \tag{41}$$

for $i < j \le N$. When P_s exists, these inequalities can be efficiently solved numerically through a linear matrix inequality method [6]. This statement gives a sufficient condition for ensuring the stability of the fuzzy system with finite α cuts, and the question that naturally arises is whether the system is stable if all of its subsystems are stable. The answer, in general, is "no" [7]. Kharitonov's theorem and related results [12] that deal with the stability of interval uncertain systems may provide an avenue to address the stability issue corresponding to the fuzzy uncertain system.

Time-Dependent System

The analysis of Sec. IV.D.1 shows that, when the system is asymptotically stable, $X(t)$ satisfies the relationship $\forall\, e < \infty$ such that $\|X(t)\| < e$, $\forall\, t > 0$, and $X(t)$ tends to 0 when $t \to \infty$, that is, $\forall\, \varepsilon > 0$, $\exists T(\varepsilon) > 0$ such that $\|X(t)\| \le \varepsilon$, $\forall\, t > T(\varepsilon)$. Therefore,

$$\begin{aligned} \int_0^\infty \|y(t)\|^2 \,\mathrm{d}t &= \int_0^\infty \|C_y X(t)\|^2 \,\mathrm{d}t \le \int_0^\infty \|C_y\|^2 \|X(t)\|^2 \,\mathrm{d}t \\ &= \int_0^T \|C_y\|^2 \|X(t)\|^2 \,\mathrm{d}t + \int_T^\infty \|C_y\|^2 \|X(t)\|^2 \,\mathrm{d}t < \infty \end{aligned} \tag{42}$$

because both the integrals on the right-hand side are bounded.

V. Illustrative Example

The design of a two-bay truss is considered to illustrate the numerical application of the fuzzy optimal control of fuzzy modeled structures. Consider a two-bay truss with four actuators as shown in Fig. 4. A fuzzy optimal controller is to be designed as a regulator to control its forced vibration response. The geometry and the material properties of the truss are assumed to have uncertain information. The total length (span) of the truss is *about* 100 in., which is divided into two equal bays. The truss is a cantilever truss with a depth of *about* 36 in. at the base and *about* 24 in. at the tip. It is assumed to undergo deflection only in the XY plane. The truss is made of aluminum with E of *about* $1 \cdot 10^7$ psi, ρ of *about* 0.1 lb/in.^{3}, and an area of cross section of each bar (or element) of *about* 0.1 in.^{2}.

For the finite element analysis of this two-dimensional truss, 6 nodes and 10 elements are used. At each node, 2 degrees of freedom are considered (at node 1, for example, q_1 and q_2 denote the 2 degrees of freedom, with q_1 denoting the vertical (Y) displacement and q_2 the horizontal (X) displacement), so that the truss will have a total of 8 degrees of freedom in the configuration space (displacements) because nodes 5 and 6 are fixed. Therefore, in the state space (displacements and velocities), there will be 16 degrees of freedom. A nonstructural mass of *about* 1.29 lb $\times$ s^2/in. per node is assigned to all of the nodes except the two fixed nodes. The feedback control system of the structure consists of four actuators located at nodes 1–4, and the sensor locations are assumed to coincide with those of the actuators. The actuator forces are to be applied only in the transverse (Y) direction of the truss, as shown in Fig. 4. The initial state is assumed to be a static displacement vector that results from the application of a 10,000-lb force at each of the actuator locations and a zero velocity vector so that the initial state vector is given by [0; 0; 0; 0; 0; 0; 0; 0; −2.64; 18.89; 2.64; 18.89; −2.51; 7.06; 2.51; 7.06].

To design the fuzzy optimal controller as a regulator, the structural model needs to be developed first. From the deterministic finite element model [13], the stiffness matrix of a fuzzified two-dimensional truss element can be expressed as

$$[K^e] = [A^e(\cdot)E^e(\cdot)/l^e](\cdot) \times \begin{bmatrix} l_{ij}(\cdot)l_{ij} & l_{ij}(\cdot)m_{ij} & -l_{ij}(\cdot)l_{ij} & -l_{ij}(\cdot)m_{ij} \\ l_{ij}(\cdot)m_{ij} & m_{ij}(\cdot)m_{ij} & -l_{ij}(\cdot)m_{ij} & -m_{ij}(\cdot)m_{ij} \\ -l_{ij}(\cdot)l_{ij} & -l_{ij}(\cdot)m_{ij} & l_{ij}(\cdot)l_{ij} & l_{ij}(\cdot)m_{ij} \\ -l_{ij}(\cdot)m_{ij} & -m_{ij}(\cdot)m_{ij} & l_{ij}(\cdot)m_{ij} & m_{ij}(\cdot)m_{ij} \end{bmatrix} \quad (43)$$

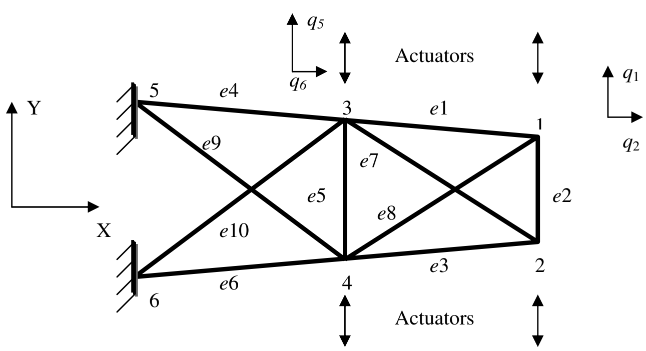

Fig. 4 A two-bay truss with four actuators.

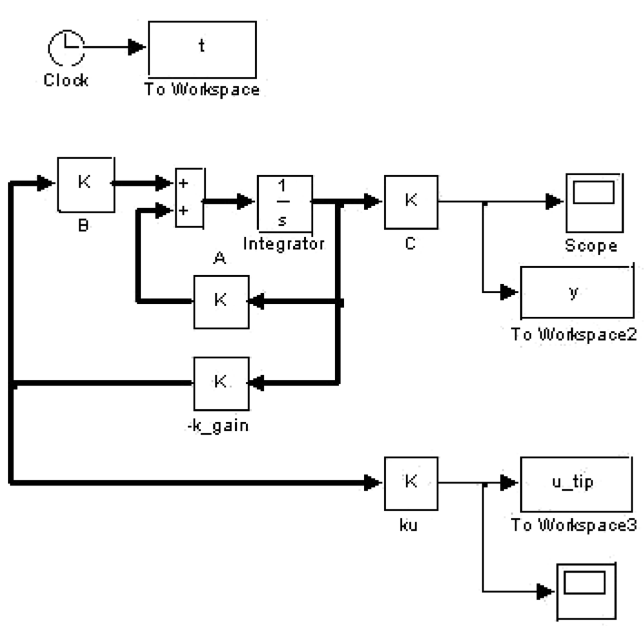

Fig. 5 Regulator design for the truss.

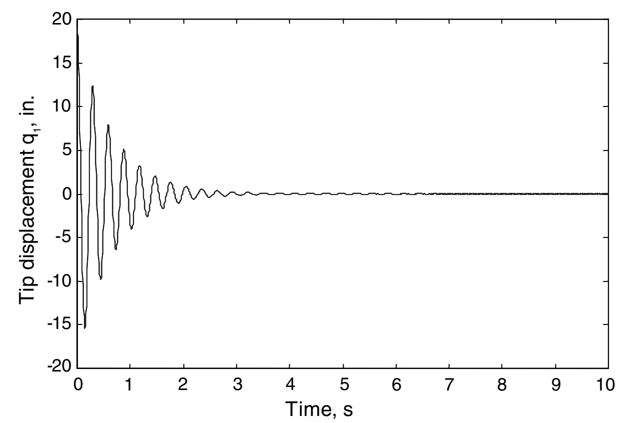

Fig. 6 Tip displacement q_1 of the truss in case 1.

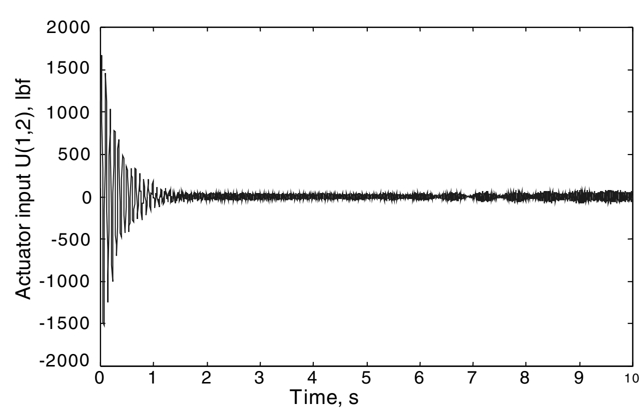

Fig. 7 Tip actuator input along q_1 in case 1.

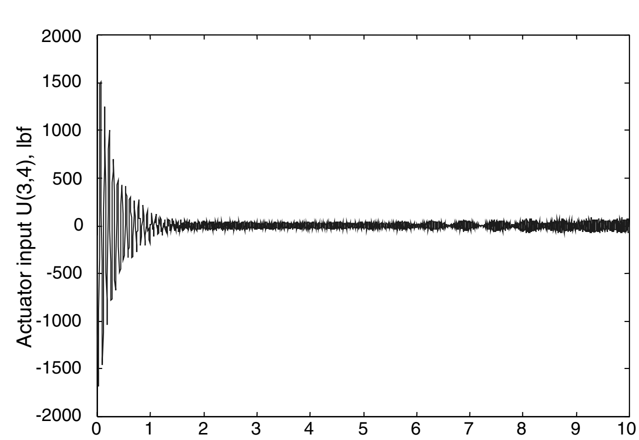

Fig. 8 Actuator input along q_5 in case 1.

where A is the area of the cross section, l is the length, and E is the Young's modulus of element e denoted by the superscript e, and $[K^e]$ is the stiffness matrix of element e in the global coordinate system. In Eq. (43), the direction cosines l_{ij} and m_{ij} can be computed as

$$l_{ij} = [X_j(-)X_i](/)l^e, \qquad m_{ij} = [Y_j(-)Y_i](/)l^e \tag{44}$$

where (X_i, Y_i) denote the global coordinates of node i, and (X_j, Y_j) denote the global coordinates of node j, with the local x axis pointing from node i to j. The mass matrix in the global coordinate system is given by

$$[M^e] = [A^e(\cdot)\rho^e(\cdot)l^e(/)6](\cdot)\begin{bmatrix} (2) & (0) & (1) & (0) \\ (0) & (2) & (0) & (1) \\ (1) & (0) & (2) & (0) \\ (0) & (1) & (0) & (2) \end{bmatrix} \tag{45}$$

where ρ^e is the mass density of element e. Thus, the structural model is defined by the fuzzy matrices given by Eqs. (43) and (45). Using the full system formulation without modal reduction, from Eqs. (6), (7), (10–12), (19), (21), and (24), the Q_i and R_i matrices can be chosen as

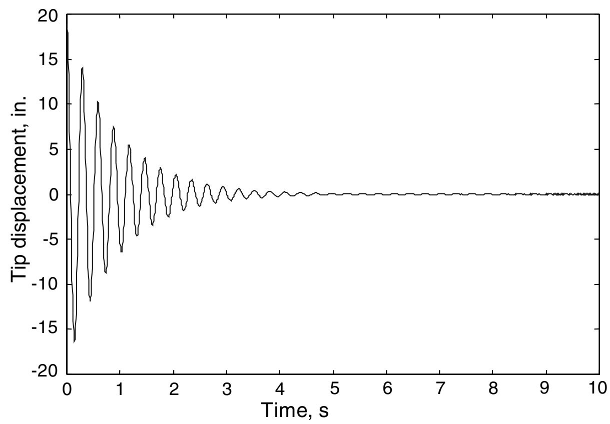

Fig. 9 Tip displacement q_1 in case 2.

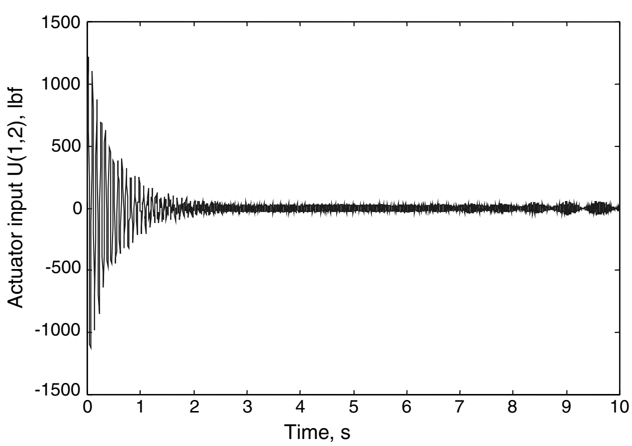

Fig. 10 Tip actuator input along q_1 in case 2.

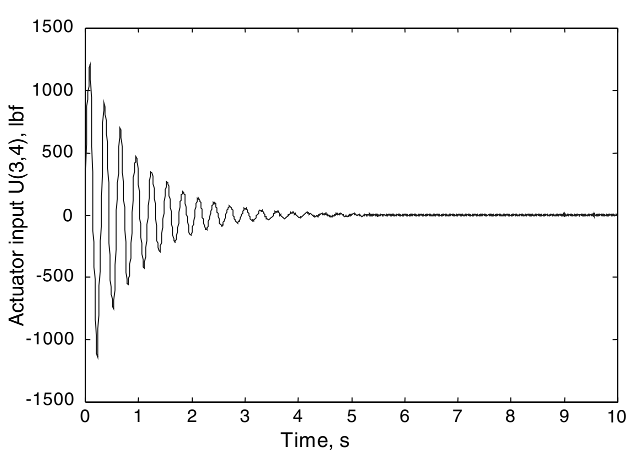

Fig. 11 Actuator input along q_5 in case 2.

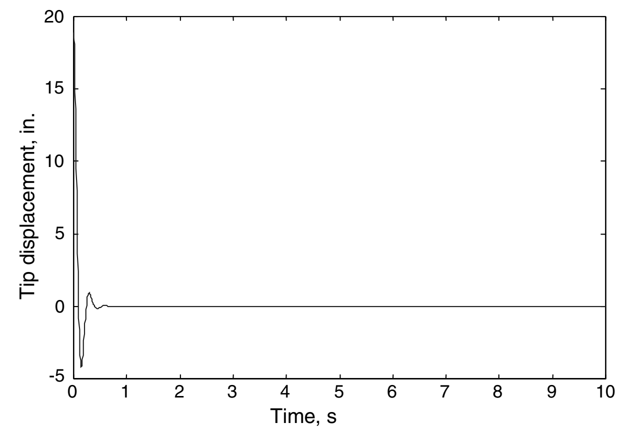

Fig. 12 Tip displacement q_1 in case 3.

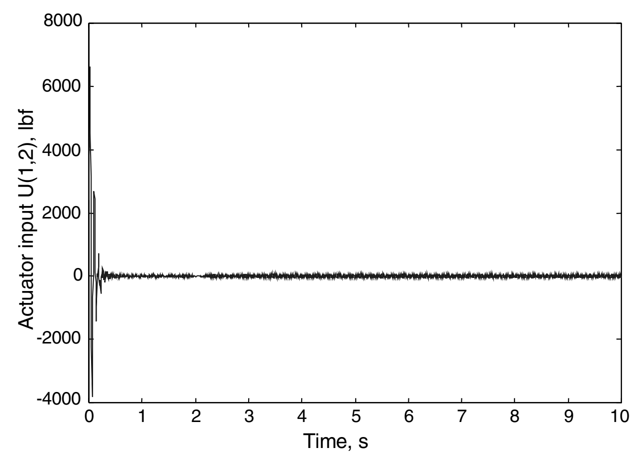

Fig. 13 Tip actuator input along q_1 in case 3.

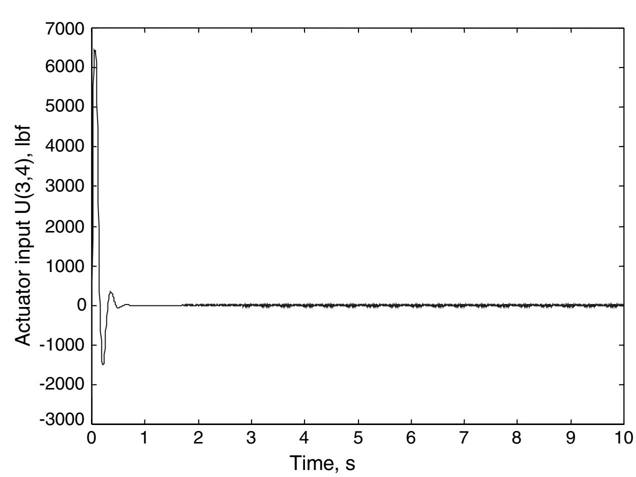

Fig. 14 Actuator input along q_5 in case 3.

Table 1 Basic parameters of the two-bay truss

α	E, psi	Area, in.^{2}	Nonstructural mass, lb $\cdot$ s^2/in.	ρ, lb/in.^{3}	length of $e1$ ($e3$)
0.0	$9.900e+006$	0.0990	1.27710	0.0990	49.50
0.2	$9.920e+006$	0.0992	1.27968	0.0992	49.60
0.4	$9.940e+006$	0.0994	1.28226	0.0994	49.70
0.6	$9.960e+006$	0.0996	1.28484	0.0996	49.80
0.8	$9.980e+006$	0.0998	1.28742	0.0998	49.90
1.0	$1.000e+007$	0.1000	1.29000	0.1000	50.00
0.8	$1.001e+007$	0.1001	1.29129	0.1001	50.05
0.6	$1.002e+007$	0.1002	1.29258	0.1002	50.10
0.4	$1.003e+007$	0.1003	1.29387	0.1003	50.15
0.2	$1.004e+007$	0.1004	1.29516	0.1004	50.20
0.0	$1.005e+007$	0.1005	1.29645	0.1005	50.25

α	length of $e2$	length of $e4$ ($e6$)	length of $e5$	length of $e7$ ($e8$)	length of $e9$ ($e10$)
0.0	23.760	49.50	29.70	56.1767	59.2363
0.2	23.808	49.60	29.76	56.2901	59.3560
0.4	23.856	49.70	29.82	56.4036	59.4756
0.6	23.904	49.80	29.88	56.5171	59.5953
0.8	23.952	49.90	29.94	56.6306	59.7150
1.0	24.000	50.00	30.00	56.7441	59.8346
0.8	24.024	50.05	30.03	56.8008	59.8945
0.6	24.048	50.10	30.06	56.8576	59.9543
0.4	24.072	50.15	30.09	56.9143	60.0141
0.2	24.096	50.20	30.12	56.9711	60.0740
0.0	24.120	50.25	30.15	57.0278	60.1338

$$Q_i = \begin{bmatrix} \theta^2 M_i & 0 \\ 0 & \theta^2 K_i \end{bmatrix}_{2n\cdot 2n}, \qquad R_i = (\theta_R)^2 b_i^T [K_i]^{-1} b_i \tag{46}$$

where θ and θ_R are constants. When the matrices Q_i and R_i, or equivalently, θ and θ_R, are changed, the amplitude of the resulting actuator inputs and the settling time of the dynamic response will also be changed. Suppose we use the full state feedback and, for each μ_i, a different set of (A_i, B_i, G_i) will be obtained. From Eqs. (21), (29), and (30), the following results can be obtained for different choices of θ and θ_R. Here $n = 8$, and the sensors are located at the same place as the actuators and the true states are the observable states. Because of symmetry, the actuators at nodes 1 and 2 have the same inputs as the actuators at nodes 3 and 4; hence, nodes 1–4 will have the same displacement responses. The regulator subsystem shown in Fig. 5 is used in this work.

The numerical results are obtained for the following sets of θ and θ_R:

Case 1: $\theta = 0.1/\sqrt{2}$, $\theta_R = 1/\sqrt{2}$

Case 2: $\theta = 0.1/\sqrt{2}$, $\theta_R = 1$

Case 3: $\theta = 1/\sqrt{2}$, $\theta_R = 1$

The time variations of the displacement q_1, the actuator input force along q_1, and the actuator input force along q_5 are shown in Figs. 6–14. It can be seen that, when θ increases, that is, when Q_i increases, the damping of the system will increase, and the settling time will decrease whereas the actuator force will be larger. When θ_R increases, that is, when R_i increases, the actuator forces will decrease and, hence, the settling time will increase. Thus, we need to make a compromise between the two performance indices, namely, the settling time and amplitude of the actuator input [14].

The gain matrix G is a $4 \cdot 16$ fuzzy matrix; Tables 1 and 2 show the basic parameters of the truss and some elements of G corresponding to the regulator design of case 3.

Next, a tracking system is designed to track the expected displacement at node 1. Let the expected value of the displacement be a crisp value of -1.00 in.. The initial conditions will be the same as [0: 0; 0; 0; 0; 0; 0; 0; -2.64; 18.89; 2.64; 18.89; -2.51; 7.06; 2.51; 7.06]. Because only one displacement component is tracked, only one integrator will be used. The tracking subsystem used is shown in Fig. 15.

The row vector C is of size 16 with all elements zero except the ninth, which is 1: $C = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]$. The initial condition for the new integrator used to track the expected displacement is 0. From Eqs. (31–33), we can choose

$$Q_i = \begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix}_{(2n+1)\cdot(2n+1)}, \qquad R_i = c[I]_{\frac{n}{2}\cdot\frac{n}{2}} \tag{47}$$

where $n = 8$. The results obtained for the case $s = 0.5 \cdot 1e4$ and $c = 1e-5$ are shown in Figs. 16–18.

From these results, we can see that when s increases, that is, when Q_i increases, the damping of the system will increase and the settling time will decrease whereas the actuator force will be larger. When c increases, that is, when R_i increases, the actuator forces will decrease and, hence, the settling time will increase. Thus, the same conclusion as in the case of the design of a regulator holds true, and we need to make a compromise between the performance indices of the settling time and the actuator input. In practice, an upper bound is placed on the actuator input to limit the total energy input. The gain matrix G is now a $4 \cdot 17$ fuzzy matrix. Figure 19 shows some elements of G (in graphical form) corresponding to the tracking system design.

Table 2 Elements of the gain matrix G (regulator design), case 3

α	$G[0][0]$	$G[1][1]$	$G[2][2]$	$G[3][3]$
0.0	210.1441	186.7538	210.2979	183.8186
0.2	210.9894	184.5566	211.0954	184.9098
0.4	211.8678	184.7858	211.9112	184.5341
0.6	212.5578	185.0209	212.5762	184.8780
0.8	213.0639	184.5608	213.0711	184.5137
1.0	213.2476	184.2096	213.2476	184.2096
0.8	213.3941	183.5794	213.3869	183.6270
0.6	213.8073	182.2693	213.7907	182.4165
0.4	214.5391	182.6638	214.5003	182.9155
0.2	215.4439	183.6282	215.3388	183.2458
0.0	212.7817	184.1597	210.4725	183.6400

α	$G[0][1]$	$G[1][2]$	$G[2][3]$	$G[3][4]$
0.0	6.3040	0.5786	-8.4040	2.2949
0.2	4.8563	0.5692	-5.5199	2.5867
0.4	4.8347	0.8260	-5.1679	3.3699
0.6	4.4325	0.7775	-4.5073	3.9315
0.8	4.0095	0.8868	-4.0157	4.1770
1.0	3.7711	0.9503	-3.7711	4.2272
0.8	3.4121	0.9920	-3.4079	4.2231
0.6	2.6022	1.0142	-2.5441	4.3209
0.4	2.2040	0.9618	-1.9020	4.8731
0.2	2.5466	1.2594	-1.9196	5.7320
0.0	5.2494	1.5458	-5.9829	7.1204

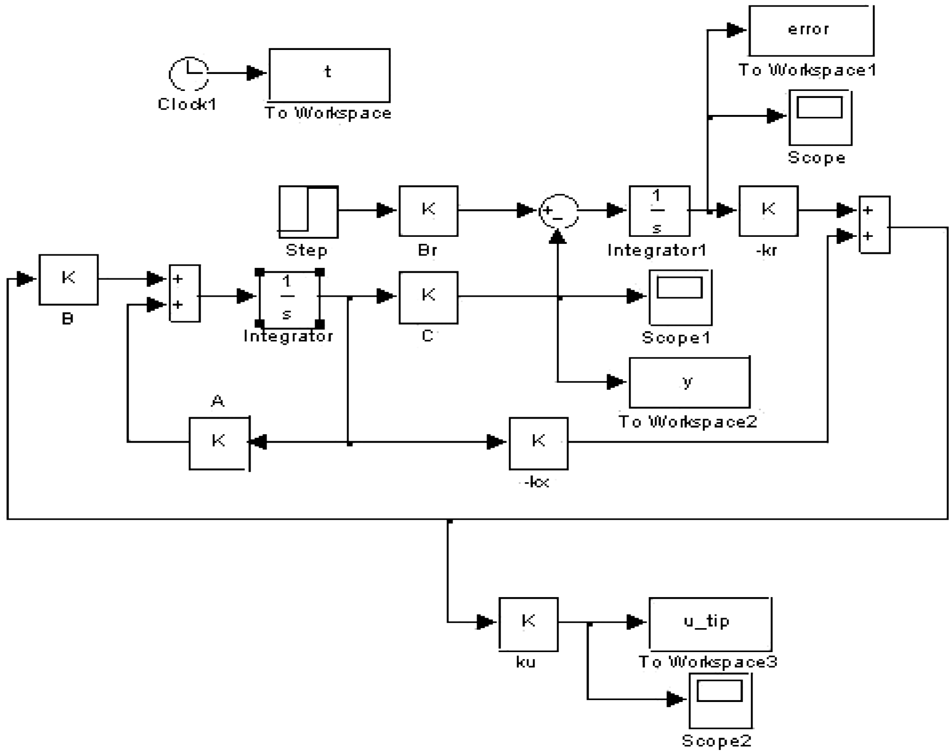

Fig. 15 Tracking subsystem design in application 1.

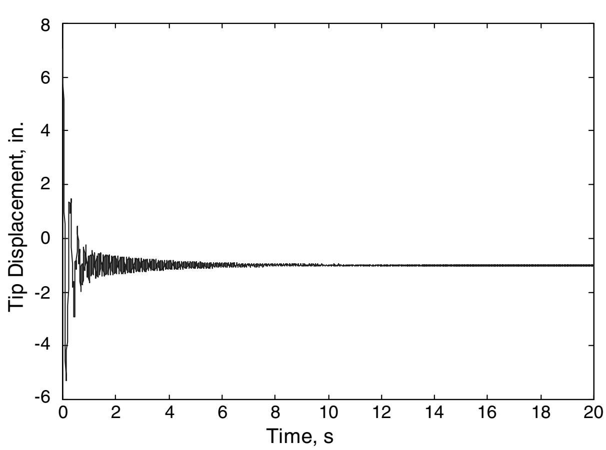

Fig. 16 Tracking tip displacement q_1.

VI. Conclusions

To handle the fuzzy uncertainties present in a structural model such as the ones obtained from a fuzzy finite element analysis, a fuzzy optimal control method is presented. First, a new model is presented based on the linear partition of the fuzzy set; for each α cut, there will be a local deterministic model, which can be viewed as a subsystem of the fuzzy system. Second, based on the new model and the deterministic linear LQR method, the global fuzzy optimal controller is obtained by fuzzily combining a series of local deterministic LQR controllers, each of which is obtained from the deterministic subsystems using the LQR method. Two situations, one with the controller as a regulator and the other as a tracking design controller, are explored. Further, the stability of the new method is investigated and it is shown that the method is asymptotically stable. When there is any damage to the structure, the dynamic response as well as the static response of the structure, such as the displacements of the structure, under a specified set of loads will be different from those under normal operating conditions (with no damage). By using the tracking system, the complete system can be made to reach a balance (such as reducing the displacements back

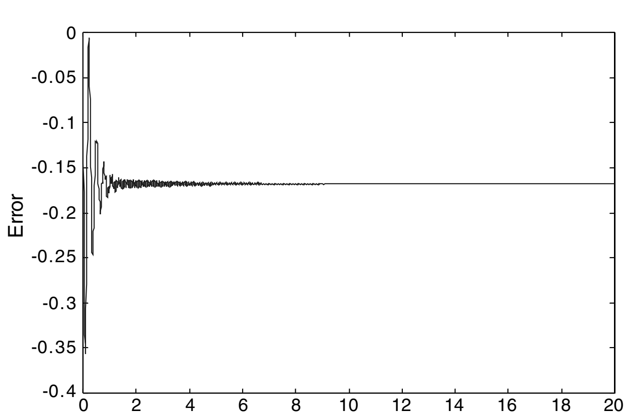

Fig. 17 Tracking error [integral of Eq. (31)].

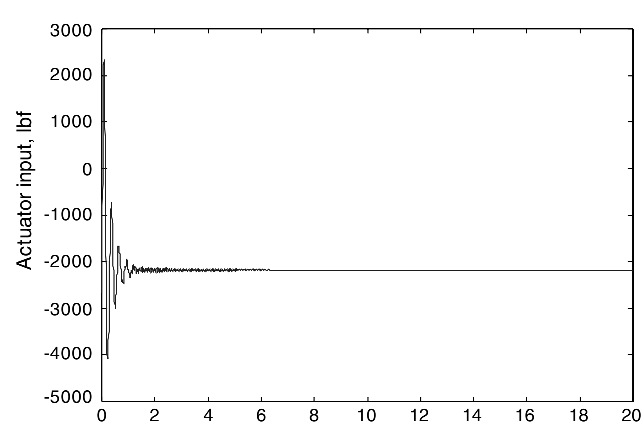

Fig. 18 Tracking tip actuator input along q_1.

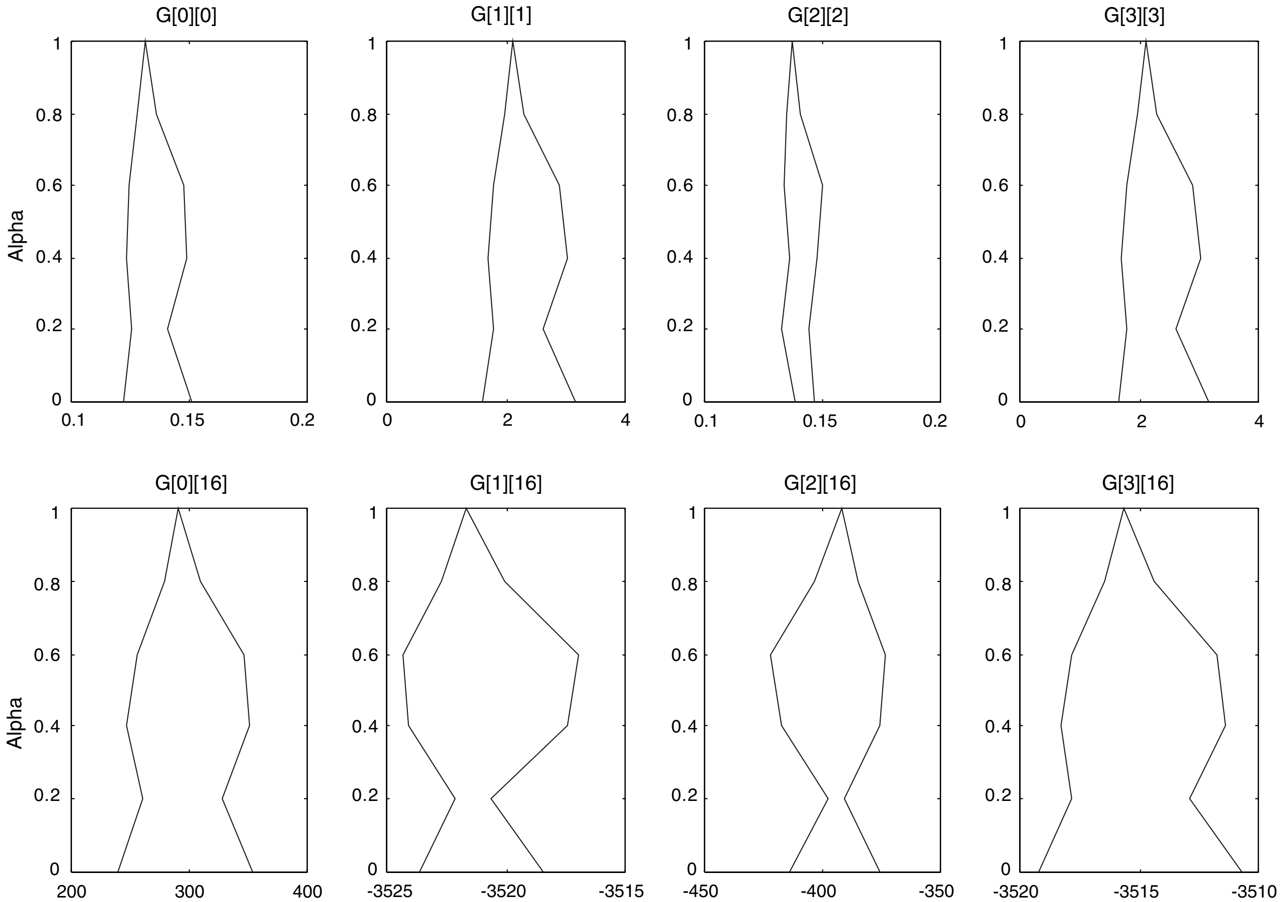

Fig. 19 Graphical representation of the elements of *G* corresponding to the tracking system design.

to within a range of their respective values under a normal state with no damage) within a finite time period. The controller design of a two-bay truss is presented to illustrate the methodology. It is also shown that, in a control system, to satisfy the goals of quicker response (such as a smaller settling time) and a bounded actuator input, one needs to make a compromise between the two.

References

[1] Takagi, T., and Sugeno, M., "Fuzzy Identification of Systems and Its Applications to Modeling and Control," *IEEE Transactions on Systems, Man, and Cybernetics*, Vol. SMC-15, No. 1, Jan.–Feb. 1985, pp. 116–132.

[2] Wang L.-X., "Stable and Optimal Fuzzy Control of Linear Systems," *IEEE Transactions on Fuzzy Systems*, Vol. 6, No. 1, Feb. 1998, pp. 137–143.
doi:10.1109/91.660813

[3] Zak, S. H., "Stabilizing Fuzzy System Models Using Linear Controllers," *IEEE Transactions on Fuzzy Systems*, Vol. 7, No. 2, April 1999, pp. 236–240.
doi:10.1109/91.755404

[4] Ma, X.-J., Sun, Z.-Q., and He, Y.-Y., "Analysis and Design of Fuzzy Controller and Fuzzy Observer," *IEEE Transactions on Fuzzy Systems*, Vol. 6, No. 1, Feb. 1998, pp. 41–51.
doi:10.1109/91.660807

[5] Tanaka, K., Ikeda, T., and Wang, H. O., "Robust Stabilization of a Class of Uncertain Nonlinear Systems via Fuzzy Control: Quadratic Stability, H^∞ Control Theory, and Linear Matrix Inequalities," *IEEE Transactions on Fuzzy Systems*, Vol. 4, No. 1, Feb. 1996, pp. 1–13.
doi:10.1109/91.481840

[6] Tanaka, K., Ikeda, T., and Wang, H. O., "Fuzzy Regulators and Fuzzy Observers: Relaxed Stability Conditions and LMI-Based Designs," *IEEE Transactions on Fuzzy Systems*, Vol. 6, No. 2, May 1998, pp. 250–265.
doi:10.1109/91.669023

[7] Wu, S.-J., and Lin, C.-T., "Optimal Fuzzy Controller Design: Local Concept Approach," *IEEE Transactions on Fuzzy Systems*, Vol. 8, No. 2, April 2000, pp. 171–185.
doi:10.1109/91.842151

[8] Wu, S.-J., and Lin, C.-T., "Optimal Fuzzy Controller Design in Continuous Fuzzy System: Global Concept Approach," *IEEE Transactions on Fuzzy Systems*, Vol. 8, No. 6, Dec. 2000, pp. 713–729.
doi:10.1109/91.890330

[9] Burl, J. B., *Linear Optimal Control: H_2 and H_∞ Methods*, Addison Wesley Longman, Reading, MA, 1999.

[10] Arnold, K., Madan, M. G., *Introduction to Fuzzy Arithmetic Theory and Application*, International Thomson Computer Press, Boston, 1991.

[11] Guran, A., and Inman, D. J., *Smart Structures, Nonlinear Dynamics, and Control*, Prentice–Hall, Upper Saddle River, NJ, 1995.

[12] Barmish, B. R., *New Tools for Robustness of Linear Systems*, Macmillan, New York/Indianapolis, IN, 1994.

[13] Rao, S. S., *The Finite Element Method in Engineering*, 4th ed., Elsevier Butterworth Heinemann, Boston, 2005.

[14] Qing, L., "Fuzzy Approach for Analysis and Optimal Control of Composite Structures with Application to Damage Detection and Health Monitoring," Ph.D. Dissertation, University of Miami, Coral Gables, FL, Dec. 2002.

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